Do we really need integral?

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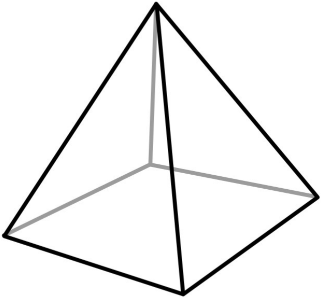
Part I

Introduction - volume and area

Volume of a cone

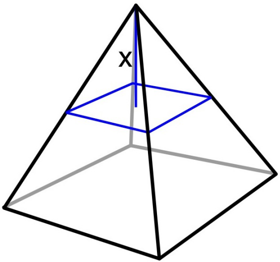


The volume formula of a cone is

$$\frac{1}{3}Ah$$

where A is the area of the base and h is the height of the cone.

Volume of a cone



If we denote the distance from the apex by x, then the area of the cross section is $A\left(\frac{x}{h}\right)^2$. By the cross-section method,

$$\text{volume } = \int_0^h A\left(\frac{x}{h}\right)^2 \, dx = \left[A\frac{1}{3}\frac{x^3}{h^2}\right]_0^h = \frac{1}{3}Ah.$$

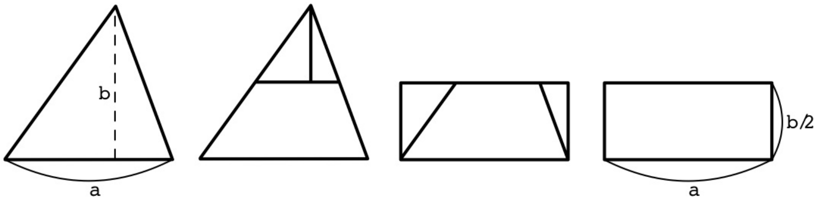
Question

Is there any simpler method to prove the formula than integral?

Area of a triangle

For a triangle, the area formula is $\frac{1}{2}ab$, where a is the length of the base, b is the height.

To show this formula, nobody uses integral!



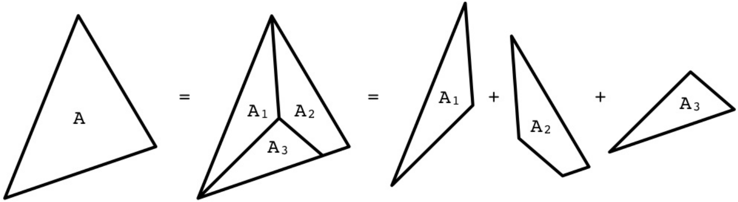
area of the triangle = area of the rectangle = $\frac{1}{2}ab$.

Properties of the area

Here are several simple properties of the area:

- The area is a nonnegative real number.
- The area of a point or a line segment is zero.
- The area of a rectangle with width a and height b is ab.
- If we cut a polygon A into n pieces $A_1, A_2, \cdots, A_n$, then

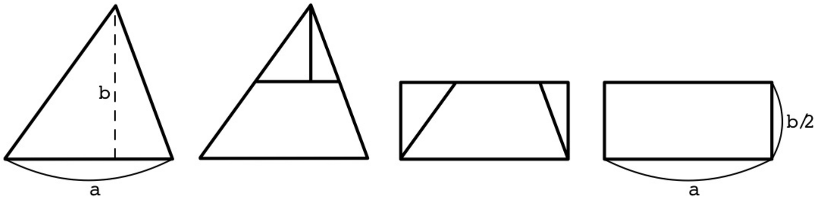
$$\text{area}(A) = \text{area}(A_1) + \text{area}(A_2) + \cdots + \text{area}(A_n).$$



Scissors-congruence

Definition

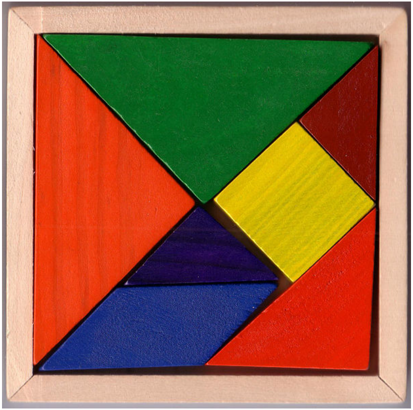
Two polygons P and Q are called scissors-congruent if P can be decomposed into finitely many polygonal pieces $P_1, P_2, \cdots, P_n$ which can be reassembled to make Q.



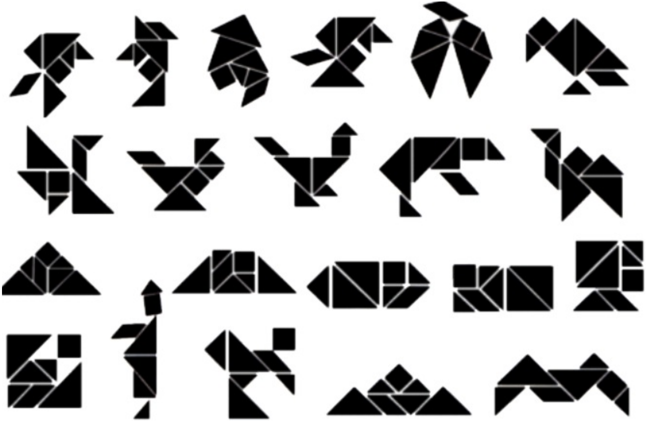
Two scissors-congruent polygons have the same area.

Scissors-congruence

Example: Tangram

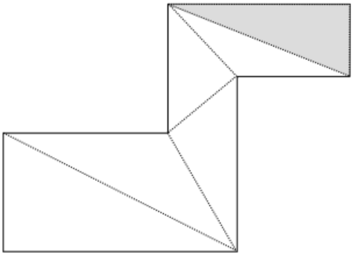


Scissors-congruence



Area of polygons

- Any polygon can be decomposed into triangles.



- A triangle is scissors-congruent to a rectangle, so we can compute its area.
- The area of the polygon is the sum of areas of triangles.

We can evaluate the area of every polygon without using calculus!

Properties of volume

- The volume is a nonnegative real number.
- The volume of a point, a line segment, or a polygon is zero.
- The area of a rectangular box with length a, width b, and height c is abc.
- If we cut a polyhedron P into n pieces $P_1, P_2, \cdots, P_n$, then

$$\mathrm{vol}(P) = \mathrm{vol}(P_1) + \mathrm{vol}(P_2) + \cdots + \mathrm{vol}(P_n).$$

- We can define the scissors-congruence for polyhedra.
- Two scissors-congruent polyhedra have the same volume.

Hilbert's question

In 1900 ICM, Hilbert asked 23 important mathematical questions (including Riemann's hypothesis, continuum hypothesis, sphere packing problem). One of them is:

Question (Hilbert, 1900)

Are any two polyhedra P and Q with same volume scissors-congruent?

Hilbert's question

Question (Hilbert, 1900)

Are any two polyhedra P and Q with same volume scissors-congruent?

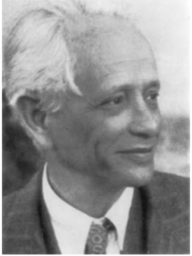
This question is important because if it is true, then we don't need to take integral to evaluate the volume of a polyhedron!

Dehn's answer

Theorem (Dehn, 1900)

There are two polyhedra P and Q with same volume, which are not scissors-congruent.

In particular, there are some polyhedra which are not scissors-congruent to a rectangular box. Therefore, we have to use integral to evaluate the volume of some polyhedra.



Dehn's answer

Theorem (Dehn, 1900)

There are two polyhedra P and Q with same volume, which are not scissors-congruent.

Wait, but... how can we prove the non-congruence (in other words, impossibility of common decomposition) of two polyhedra?

Part II

Linear maps

A functional equation

Consider a function $f : \mathbb{R} \to \mathbb{R}$ satisfying following functional equation:

$$f(x+y) = f(x) + f(y).$$

Then f has several interesting properties.

- $f(0) = f(0+0) = f(0) + f(0) \Rightarrow f(0) = 0$
- $f(2x) = f(x+x) = f(x) + f(x) = 2f(x) \Rightarrow f(2x) = 2f(x)$
- More generally, for any $n \in \mathbb{N}$, $f(nx) = nf(x)$.
- $f(x) + f(-x) = f(x + (-x)) = f(0) = 0 \Rightarrow f(-x) = -f(x)$
- $f(x) = f(\frac{1}{2}x + \frac{1}{2}x) = f(\frac{1}{2}x) + f(\frac{1}{2}x) = 2f(\frac{1}{2}x) \Rightarrow f(\frac{1}{2}x) = \frac{1}{2}f(x)$
- More generally, for any $m \in \mathbb{N}$, $f(\frac{1}{m}x) = \frac{1}{m}f(x)$.
- We have $f(\frac{n}{m}x) = \frac{n}{m}f(x)$ for every rational number $\frac{n}{m}$.

A functional equation $f(x+y) = f(x) + f(y)$

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is called a $\mathbb{Q}$-linear map if
$f(x+y) = f(x) + f(y)$ (so $f(qx) = qf(x)$ for all $q \in \mathbb{Q}$).

Q. Why is it called a linear map?

Example 1: A function $f(x) = cx$ for some constant c has the property.

$$f(x+y) = c(x+y) = cx + cy = f(x) + f(y)$$

A. The graph is linear!

A functional equation $f(x+y) = f(x) + f(y)$

But there are more examples!

If we know $f(1)$, we know $f(\frac{2014}{2013})$ because $f(\frac{2014}{2013}) = \frac{2014}{2013} f(1)$.

Q. If $f(1) = 5$, what is $f(3)$? What is $f(\frac{2}{3})$?

Q. What is $f(\sqrt{2})$?

A functional equation $f(x+y) = f(x) + f(y)$

But there are more examples!

If we know $f(1)$, we know $f(\frac{2014}{2013})$ because $f(\frac{2014}{2013}) = \frac{2014}{2013} f(1)$.

Q. If $f(1) = 5$, what is $f(3)$? What is $f(\frac{2}{3})$?

Q. What is $f(\sqrt{2})$?

There is no problem even if we define $f(\sqrt{2}) = 100$.

$f(\frac{2}{3}) = \frac{10}{3}$, $f(\frac{3}{7}\sqrt{2}) = \frac{300}{7}$

Q. Is it possible to have a $\mathbb{Q}$-linear map f with $f(1) = 1$, $f(\sqrt{2}) = 3$?

Q. Is there a $\mathbb{Q}$-linear map f with $f(1) = 2$, $f(\sqrt{5}) = \pi$, $f(\frac{\sqrt{5}}{6}) = 0$?

Summary: If we define $f(a)$, then for any $q \in \mathbb{Q}$, $f(qa)$ is determined.
Except it, we can freely choose $f(b)$ for $b \neq qa$ with $q \in \mathbb{Q}$.

Part III

Dehn's theorem

Dehn invariant

Let $f : \mathbb{R} \to \mathbb{R}$ be a function with following two properties:

- (f is $\mathbb{Q}$-linear) $f(x+y) = f(x) + f(y)$;
- $f(\pi) = 0$.

For a polynomial P with edges e_1, e_2, $\cdots$, e_n, we define

- $\ell(e_i)$ as the length of the edge e_i,
- $\theta(e_i)$ as the dihedral angle between the faces meeting at e_i.

Definition

Let P be a polyhedron with edges e_1, e_2, $\cdots$, e_n. The Dehn invariant $D_f(P)$ with respect to f is defined by

$$D_f(P) = \sum_{i=1}^{n} \ell(e_i) f(\theta(e_i)).$$

Example - a rectangular box

Let C be a rectangular box.

There are 12 edges $e_1, e_2, \cdots, e_{12}$.

Then $\theta(e_1) = \theta(e_2) = \cdots = \theta(e_{12}) = \frac{\pi}{2}$.

Therefore

$$\begin{aligned} D_f(C) &= \sum_{i=1}^{12} \ell(e_i) f(\theta(e_i)) = \sum_{i=1}^{12} \ell(e_i) \cdot f(\frac{\pi}{2}) \\ &= \sum_{i=1}^{12} \ell(e_i) \cdot \frac{1}{2} f(\pi) = \sum_{i=1}^{12} \ell(e_i) \cdot 0 = 0. \end{aligned}$$

Note that this is true for any $\mathbb{Q}$-linear map f with $f(\pi) = 0$.

Dehn's theorem

Theorem (Dehn, 1900)

If a polyhedron P is cut into n polyhedra P_1, P_2, $\cdots$, P_n, then $D_f(P) = D_f(P_1) + D_f(P_2) + \cdots + D_f(P_n)$.

Corollary

For two scissors-congruent polyhedra P and Q, $D_f(P) = D_f(Q)$.

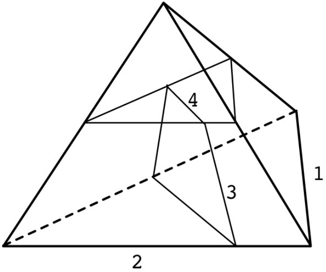
Corollary

If P is a polyhedron which is scissors-congruent to a rectangular box, then $D_f(P) = 0$ for every $\mathbb{Q}$-linear map f with $f(\pi) = 0$.

So if there is a polyhedron with $D_f(P) \neq 0$, then P is not scissors-congruent to a rectangular box!

Dehn's theorem - proof

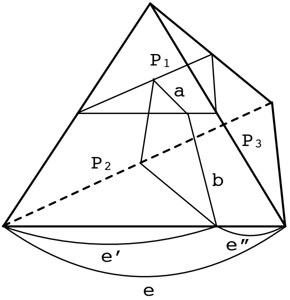
If we cut a polyhedron P into many polyhedra, there are four possible changes on the set of edges.



1. Nothing happens.
2. An edge is cut into two edges.
3. A new edge forms on a face of P.
4. A new edge forms on the interior of P.

Dehn's theorem - proof

Case 2: An edge is cut into two edges.



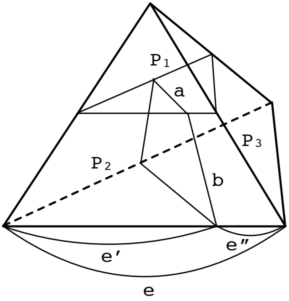
Look at e, e', and e''.

Note that $\theta(e) = \theta(e') = \theta(e'')$.

$$\begin{aligned}\ell(e)f(\theta(e)) &= (\ell(e') + \ell(e''))f(\theta(e)) = \ell(e')f(\theta(e)) + \ell(e'')f(\theta(e)) \\ &= \ell(e')f(\theta(e')) + \ell(e'')f(\theta(e''))\end{aligned}$$

Dehn's theorem - proof

Case 3: A new edge forms on a face of P.



See b on the picture.

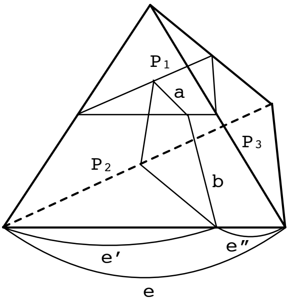
There are two polyhedra P_2 and P_3 share the edge b.

Let $\theta_2(b)$ (resp. $\theta_3(b)$) be the dihedral angle of P_2 (resp. P_3) at b.

$$\begin{aligned}\ell(b)f(\theta_2(b)) + \ell(b)f(\theta_3(b)) &= \ell(b)(f(\theta_2(b)) + f(\theta_3(b))) \\ &= \ell(b)(f(\theta_2(b) + \theta_3(b))) = \ell(b)f(\pi) = 0\end{aligned}$$

Dehn's theorem - proof

Case 4: A new edge forms on the interior of P.



See a on the picture.

The sum of all dihedral angles at a is 2π. Therefore $f(2\pi) = 0$ and by a similar idea, the sum of $\ell(a)f(\theta(a))$'s is zero.

Dehn's theorem - proof

When we compute $D_f(P_1) + D_f(P_2) + \cdots + D_f(P_n)$, ...

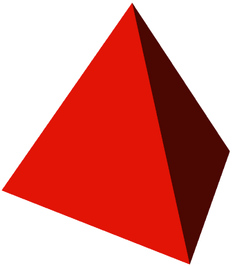
- We don't need to compute $\ell(e)f(\theta(e))$ for newly formed edges (cases 3 and 4), because the sum of $\ell(e)f(\theta(e))$ for such e is zero.
- For a decomposed edge e into e_1 and e_2, then $\ell(e_1)f(\theta(e_1)) + \ell(e_2)f(\theta(e_2)) = \ell(e)f(\theta(e))$.

As a conclusion, $D_f(P_1) + D_f(P_2) + \cdots + D_f(P_n)$ is equal to the sum of $\ell(e)f(\theta(e))$ for original edges, which is $D_f(P)$.

A calculation of Dehn's invariant

So it is sufficient to find a polyhedron P with $D_f(P) \neq 0$!

Let P be a regular tetrahedron.



There are 6 edges with length ℓ, dihedral angle α.

Then $\cos\alpha = \frac{1}{3}$.

A calculation of Dehn's invariant

Claim: α is not a rational multiple of π.

Sketch of proof:

- Show that $\cos(k+1)\alpha = 2\cos k\alpha\cos\alpha - \cos(k-1)\alpha$.
- Use induction to show $\cos k\alpha = \frac{A_k}{3^k}$ where $A_k \in \mathbb{Z}$ and $3 \nmid A_k$.
- If $\alpha = \frac{p}{q}\pi$, then $\cos q\alpha = \cos p\pi = \pm 1$.
- But $\cos q\alpha = \frac{A_q}{3^q}$, so $A_q = \pm 3^q$, which is a multiple of 3.

  Contradiction.

A calculation of Dehn's invariant

α is not a rational multiple of π.

So we may find a $\mathbb{Q}$-linear map $f : \mathbb{R} \to \mathbb{R}$ such that $f(\pi) = 0$ and $f(\alpha) = 1$.

Then

$$D_f(P) = \sum_{i=1}^{6} \ell f(\alpha) = 6(\ell \cdot 1) = 6\ell \neq 0.$$

Therefore a regular tetrahedron is not scissors-congruent to a rectangular box. In other words, it is impossible to compute the volume without using integral!

By a similar idea, we are able to show that a pyramid Q over a square base whose width and height ℓ has $D_f(Q) \neq 0$.

Part IV

. . . and more

Dehn-Sydler theorem

Question

When are two polyhedra P and Q scissors-congruent?

Obvious restrictions:

- $\mathrm{vol}(P) = \mathrm{vol}(Q)$.
- $D_f(P) = D_f(Q)$.

That's all!

Theorem (Sydler, 1965)

If $\mathrm{vol}(P) = \mathrm{vol}(Q)$ and $D_f(P) = D_f(Q)$ for every f, then P and Q are scissors-congruent.

A trend of geometry in 20th century - Algebraization

Question

For two geometric objects X and Y, how can we prove $X \neq Y$?

Answer using algebra:

Step 1. For each object X, construct an algebraic object $F(X)$ with the property that $X = Y \Rightarrow F(X) = F(Y)$.

Step 2. Compute $F(X)$ and $F(Y)$.

Step 3. If $F(X) \neq F(Y)$, $X \neq Y$!

Because algebraic objects are relatively easy to study, in many cases, Step 2 is doable.

A trend of geometry in 20th century - Algebraization

In 20th century geometry, there are tons of such examples!

symmetry group, fundamental group, homotopy group, homology group, cohomology ring, holonomy group, Chern classes, derived category, K-group, motive, Alexander polynomial, Jones polynomial, Fukaya category, Gromov-Witten invariants, quantum cohomology, $\cdots$

Thank you!