MATH 3005 MIDTERM EXAM 2 SOLUTION

SPRING 2014 - MOON

Write your answer neatly and show steps.
Any electronic devices including calculators, cell phones are not allowed.

(1) Write the definition.

(a) (3 pts) An even permutation.

A permutation σ is called an even permutation, if it is a product of even number of 2-cycles.

(b) (3 pts) An isomorphism of groups.

For two groups G and $\overline{G}$, a map $\phi : G \to \overline{G}$ is called an isomorphism if ϕ is one-to-one, onto and for all $x, y \in G$, $\phi(xy) = \phi(x)\phi(y)$.

(c) (3 pts) For a permutation group G on a set S and $a \in S$, the orbit of a.

The orbit $\mathrm{orb}_a(G)$ is the set $\{\phi(a) \mid \phi \in G\}$.

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(2) Let $\alpha = (134562)$, $\beta = (1243)(56)$ be two permutations in S_6.

(a) (3 pts) Compute β^{-1} as a product of disjoint cycles.

$$\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{bmatrix} \Rightarrow \beta^{-1} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 2 & 6 & 5 \end{bmatrix}$$
$$\Rightarrow \beta^{-1} = (1342)(56)$$

(b) (3 pts) Compute β^{-1} as a product of two cycles.

$$\beta^{-1} = (1342)(56) = (12)(14)(13)(56)$$

(c) (4 pts) Compute $\beta\alpha\beta^{-1}$ as a product of disjoint cycles.

$$\beta\alpha\beta^{-1} = (1243)(56)(134562)(1342)(56) = (136542)$$

- If (a) is incorrect, then -1 pt from (b) and (c).

(3) (8 pts) Prove that D_6 is *not* isomorphic to A_4.

D_6 has an element of order 6, a rotation by $60°$. On the other hand, an even permutation σ in A_4 is of the form $(ab)(cd)$ or (abc). So its order is 2 or 3. Therefore D_6 is not isomorphic to A_4.

- Indicating D_6 has an order 6 element: 3 pts.
- Mentioning that A_4 doesn't have an order 6 element: 3 pts.
- Explaining why A_4 has no order 6 element: 2 pts.

(4) (8 pts) Let H be a subgroup of G. Let $a, b \in G$. Prove that $aH = bH$ if and only if $a \in bH$.

Suppose that $aH = bH$. Then $a = ae \in aH = bH$.

Conversely, suppose that $a \in bH$. Then $a = bh$ for some $h \in H$. For any $ak \in aH$ for some $k \in H$. So $ak = bhk \in bH$ and therefore $aH \subset bH$. Because $b = ah^{-1}$, for any $bt \in bH$ with $t \in H$, $bt = ah^{-1}t \in aH$ and $bH \subset aH$. Therefore we have $aH = bH$.

- Showing that $aH = bH$ implies $a \in bH$: 3 pts.
- Proving that $aH \subset bH$ when $a \in bH$: 2 pts.
- Showing that $bH \subset aH$ when $a \in bH$: 3 pts.

(5) Let G be a group of order 55.

(a) (4 pts) List all possible orders of a subgroup of G.

For $H \leq G$, by Lagrange's theorem, $|H|||G| = 55$. So $|H| = 1, 5, 11, 55$.

- Excluding 1 or 55: -1 pt.

(b) (5 pts) Suppose that H, K are two distinct subgroups of G of order 11. Show that $H \cap K = \{e\}$.

$H \cap K$ is a subgroup of H. By Lagrange's theorem, $|H \cap K|||H| = 11$. So $|H \cap K| = 1$ or 11. If $|H \cap K| = 11 = |K| = |H|$, then $K = H \cap K = H$. Because H and K are two distinct subgroups, $|H \cap K| = 1$ so $H \cap K = \{e\}$.

(c) (6 pts) By using (b), show that there is an element $a \in G$ of order 5.

Pick a nonidentity $x \in G$. Then $|x| = 5, 11, 55$. If $|x| = 5$, x is what we want. If $|x| = 55$, then G is cyclic and x^{11} is an order 5 element.

Sol 1. Suppose that there is no order 55 element. If H and K are two distinct subgroups of order 11, then $|HK| = \dfrac{|H||K|}{|H \cap K|} = \dfrac{11^2}{1} = 121$. But $|HK| \leq |G| = 55$ so it is impossible. Therefore there is only one subgroup H of order 11. If we take $x \in G - H$, then $|x| = 5$ because if not, $|x| = 11$ and $\langle x \rangle$ is a subgroup of order 11 which is different from H.

Sol 2. If $|x| = 11$, then $\langle x \rangle$ is an order 11 subgroup of G. Because the intersection of any two order 11 subgroups is only the identity, the number of order 11 elements is $10k + 1$ for some $k \in \mathbb{N}$. Since 55 is not of the form $10k + 1$, there must be an order 5 element.

(6) (10 pts) Four dwarves and four elves are going to discuss how to protect middle earth from the attack of Sauron. On a round table with eight seats, how many distinguishable ways can they be seated?

Let S be the set of all arrangements of four dwarves and four elves on the table. Then the number of elements of S is $\binom{8}{4} = 70$. Because if one takes a rotation of the table, then we obtain an indistinguishable arrangement, a permutation group G isomorphic to $\mathbb{Z}_8$ permutes S. Let $G = \langle R_\theta \rangle$, where R_θ is a counterclockwise rotation by $\theta = \frac{360}{8}^\circ = 45^\circ$.

For id, $\mathrm{fix}(\mathrm{id}) = S$ and $|\mathrm{fix}(\mathrm{id})| = 105$. For R_θ, there is no fixed arrangement because for a fixed arrangement, all seats has to be assigned for dwarves only or elves only. So $|\mathrm{fix}(R_\theta)| = 0$. By the same reason, $|\mathrm{fix}(R_\theta^3)| = |\mathrm{fix}(R_\theta^5)| = |\mathrm{fix}(R_\theta^7)| = 0$.

For $R_\theta^2 = R_{2\theta}$, there are two fixed arrangements, which are alternating seatings. So $|\mathrm{fix}(R_\theta^2)| = 2$. Similarly, $|\mathrm{fix}(R_\theta^6)| = 2$ also.

For $R_\theta^4 = R_{4\theta}$, a fixed arrangement is that the opposite seat is possessed by the same tribe. So it suffices to assign the half of the table. Therefore $|\mathrm{fix}(R_\theta^4)| = \binom{4}{2} = 6$.

In summary,

$$\frac{1}{|G|}\sum_{\phi \in G} |\mathrm{fix}(\phi)| = \frac{1}{8}(70 + 0 + 2 + 0 + 6 + 0 + 2 + 0) = 10.$$

Therefore there are 10 distinguishable seatings.

- Stating Burnside's theorem: 2 pts.
- Indicating that the permutation group on S is $\mathbb{Z}_8$: 4 pts.
- Evaluating the number of all assignments 70: 6 pts.
- Computing the numbers of elements in fixed point sets: +1 pts each.
- Getting the answer 10: 10 pts.