MATH 1207 R02 MIDTERM EXAM 1 SOLUTION

SPRING 2016 - MOON

- Write your answer neatly and show steps.
- Except calculators, any electronic devices including laptops and cell phones are not allowed.

(1) Quick survey.

(a) (1 pt) This class is:

Too easy		Moderate				Too difficult
1	2	3	4	5	6	7

(b) (2 pts) Write any suggestion for improving this class. (For instance, give more examples in class, explain proofs in detail, give more homework, slow down the tempo, ...)

(2) (5 pts) Find the average of $y = \sin x$ over the interval $[0, \pi]$.

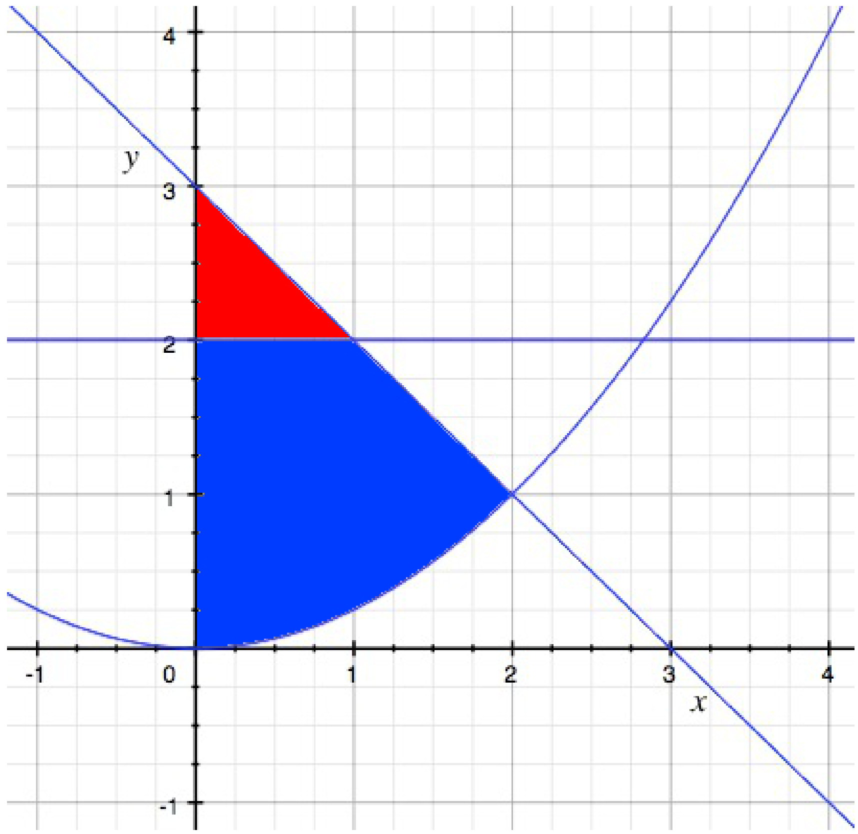
$$\int_0^\pi \sin x dx = -\cos x]_0^\pi = -\cos \pi - (-\cos 0) = 1 - (-1) = 2$$

$$\text{Average } = \frac{1}{\pi - 0}\int_0^\pi \sin x dx = \frac{2}{\pi}$$

- Writing the formula $\frac{1}{\pi - 0}\int_0^\pi \sin x dx$ for the average: 3 pts.
- Getting the answer $\frac{2}{\pi}$: 5 pts.

Date: March 7, 2015.

(3) (a) (3 pts) Sketch the region bounded by $y = -x + 3$, $y = 2$, $y = \dfrac{x^2}{4}$ and y-axis.



The blue region is the region bounded by given equations.

- Sketching curves: 1 pt.
- Indicating the region correctly: 3 pts.

(b) (6 pts) Find the area of the region in (a).

To find the area of the blue region, it is sufficient to subtract the area of the red triangle from the area between two curves $y = -x + 3$ and $y = \dfrac{x^2}{4}$.

$$\text{Area of } {\color{red}\blacksquare} \text{ and } {\color{blue}\blacksquare} = \int_0^2 -x + 3 - \frac{x^2}{4} dx = -\frac{1}{2}x^2 + 3x - \frac{x^3}{12}\Big]_0^2$$

$$= -\frac{4}{2} + 6 - \frac{8}{12} = \frac{10}{3}$$

$$\text{Area of } {\color{blue}\blacksquare} = \frac{10}{3} - \frac{1}{2} = {\color{red}\frac{17}{6}}$$

- Setting up an integral describing the area: 3 pts.
- Getting the answer ${\color{red}\frac{17}{6}}$: 6 pts.

(4) (6 pts) The great Pyramid of King Khufu was built around 2560 BC. Its base is a square with side length 756 ft, and its height is 481 ft. Find the volume of it using integral. (Using the volume formula of a cone is not allowed.)

Let x be the height (vertical distance) measured from the apex of Pyramid. The section perpendicular to the vertical axis is a square whose side length $s(x)$ is proportional to x. Therefore we have

$$\frac{s(x)}{x} = \frac{756}{481},$$

or equivalently,

$$s(x) = \frac{756}{481}x.$$

The area of the section is

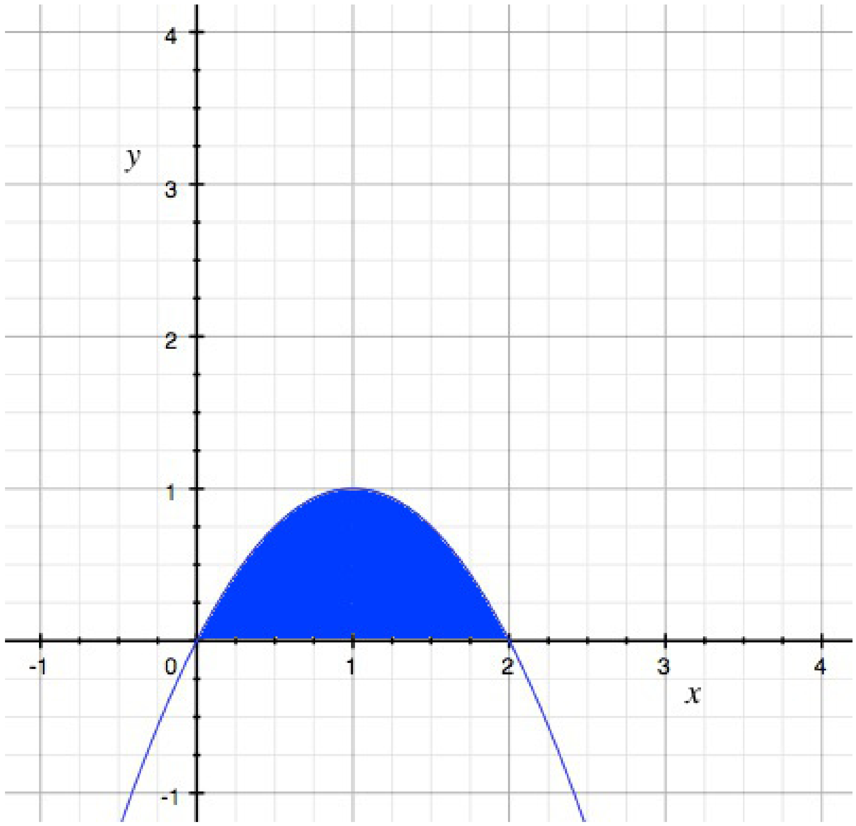
$$s(x)^2 = \left(\frac{756}{481}x\right)^2 = \frac{756^2}{481^2}x^2.$$

$$\begin{aligned}\text{Volume} &= \int_0^{481} \frac{756^2}{481^2}x^2dx = \frac{756^2}{481^2} \cdot \frac{1}{3}x^3\Big]_0^{481} \\ &= \frac{756^2}{481^2} \cdot \frac{1}{3} \cdot 481^3 = \frac{756^2 \cdot 481}{3} = 91636272\end{aligned}$$

Therefore the volume of Pyramid is 91,636,272 ft^3.

- Finding the side length $\frac{756}{481}x$ of the section: +2 pts.
- Setting up the volume formula $\frac{756^2}{481^2}x^2dx$: +2 pts.
- Getting the answer 91,636,272 ft^3: +2 pts.
- Writing the answer without an appropriate unit: -1 pt.

(5) (a) (3 pts) Sketch the region bounded by the graph of $y = 2x - x^2$ and x-axis.



(b) (6 pts) Find the volume of the solid generated by rotating the planar region in (a) about x-axis.
The section of the solid of revolution is a circular disk whose radius is $2x - x^2$.

$$\begin{aligned}\text{Volume} &= \int_0^2 \pi(2x - x^2)^2 dx \\ &= \pi \int_0^2 4x^2 - 4x^3 + x^4 dx = \pi \left(\frac{4}{3}x^3 - x^4 + \frac{1}{5}x^5\right)\Big]_0^2 \\ &= \pi(\frac{32}{3} - 16 + \frac{32}{5}) = \frac{16\pi}{15}\end{aligned}$$

- Constructing the integral $\int_0^2 \pi(2x - x^2)^2 dx$ describing the volume: 4 pts.
- Getting the answer $\frac{16\pi}{15}$: 6 pts.

(c) (6 pts) Find the volume of the solid generated by rotating the planar region in (a) about y-axis.

We can apply the shell method.

$$\begin{aligned} \text{Volume} &= \int_0^2 2\pi x(2x - x^2)dx \\ &= 2\pi \int_0^2 2x^2 - x^3 dx = 2\pi \left(\frac{2}{3}x^3 - \frac{1}{4}x^4 \right) \Big]_0^2 \\ &= 2\pi \left(\frac{16}{3} - 4 \right) = \frac{8\pi}{3} \end{aligned}$$

- Giving the integral $\int_0^2 2\pi x(2x - x^2)dx$: 4 pts.
- Obtaining the answer $\frac{8\pi}{3}$: 6 pts.

(6) (5 pts) Evaluate

$$\int_1^e \frac{(\ln x)^2}{x} dx.$$

$$u = \ln x \Rightarrow du = \frac{1}{x}dx, u(1) = \ln 1 = 0, u(e) = \ln e = 1$$

$$\int_1^e \frac{(\ln x)^2}{x} dx = \int_0^1 u^2 du = \frac{1}{3}u^3 \Big]_0^1 = \frac{1}{3}$$

- Finding an appropriate substitution $u = \ln x$: 2 pts.
- By using substitution method, getting $\int_0^1 u^2 du$: 4 pts.
- Getting the answer $\frac{1}{3}$: 5 pts.

(7) Let $f(x) = 3x + \sin x$.

(a) (2 pts) What is the domain of f?

$f(x)$ can be found for all x. So the domain is the set of all real numbers $\mathbb{R}$ (or $(-\infty, \infty)$).

(b) (3 pts) Show that f is one-to-one.

$$f'(x) = 3 + \cos x$$

Because $-1 \le \cos x \le 1$, $f'(x) \ge 2$. Therefore f is an increasing function and one-to-one.

- Finding the derivative $f'(x) = 3 + \cos x$: 1 pt.
- Getting that f is increasing from $f'(x) > 0$: 3 pts.

(c) (3 pts) Find $f^{-1}(3\pi)$.

$$f^{-1}(3\pi) = x \Leftrightarrow f(x) = 3\pi \Leftrightarrow 3x + \sin x = 3\pi$$

Because $\sin \pi = 0$, $f(\pi) = 3\pi + \sin \pi = 3\pi$. Therefore $f^{-1}(3\pi) = \pi$.

(d) (3 pts) Find $(f^{-1})'(3\pi)$.

$$(f^{-1})'(3\pi) = \frac{1}{f'(\pi)} = \frac{1}{3 + \cos \pi} = \frac{1}{3 - 1} = \frac{1}{2}$$

- Applying the inverse function theorem and getting $(f^{-1})'(3\pi) = \dfrac{1}{f'(\pi)}$: 2 pts.
- Getting the answer $\dfrac{1}{2}$: 3 pts.

(8) (6 pts) After the consumption of an alcoholic beverage, the concentration of alcohol in the bloodstream (blood alcohol concentration, or *BAC*) surges as the alcohol is absorbed, followed by a gradual decline as the alcohol is metabolized. The function

$$C(t) = 1.35te^{-2.802t}$$

models the BAC of a male, measured in mg/mL, subjects t hours after rapid consumption of 15 mL of ethanol (corresponding to one alcoholic drink). Find the maximum BAC during the first 3 hours, and indicate when it occurs.

$$C'(t) = 1.35e^{-2.802t} + 1.35te^{-2.802t} \cdot (-2.802) = 1.35e^{-2.802t}(1 - 2.802t)$$

Because $e^{-2.802t} > 0$,

$$C'(t) = 0 \Leftrightarrow 1.35e^{-2.802t}(1 - 2.802t) = 0 \Leftrightarrow 1 - 2.802t = 0 \Leftrightarrow t = \frac{1}{2.802}$$

$$C(\frac{1}{2.802}) = 1.35 \cdot \frac{1}{2.802}e^{-2.802 \cdot \frac{1}{2.802}} = 1.35 \cdot \frac{1}{2.802}e^{-1} \approx 0.1772$$

$$C(0) = 1.35 \cdot 0 \cdot e^{0} = 0$$

$$C(3) = 1.35 \cdot 3 \cdot e^{-2.802 \cdot 3} \approx 0.0009$$

Therefore the maximum BAC is 0.1772 mg/mL, and it occurs $\frac{1}{2.802} \approx 0.3569$ hours after the consumption.

- Finding the derivative $C'(t) = 1.35e^{-2.802t}(1 - 2.802t)$: 3 pts.
- Obtaining the critical point $t = \frac{1}{2.802}$: 4 pts.
- Writing the maximum BAC 0.1772 mg/mL with an appropriate unit: 6 pts.
- Omitting the unit: -1 pt.